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4×4 Parametric Integer Discrete Cosine Transforms

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***Abstract*—One of the famous transformations is discrete cosine transform (DCT), which is always used in digital image coding standards like JPEG and MPEG. DCT has different types and all of DCTs have excellent energy compaction properties. Meanwhile, matrices of DCTs II and IV are examined by a lot of researchers. However, other types of DCTs are rarely developed. Therefore, this paper presents 4×4 parametric integer DCTs, which cannot only represent DCTs II and IV but other types of DCTs (i.e. DCT I, V, VIII). It shows an excellent performance in terms of mean square errors and transform coding gains while it is comparing with state of the art.**

***Index Terms*—Image compression, discrete cosine transforms (DCTs), parametric integer transform**

I. Introduction

Transform coding, converting the data from the spatial to the frequency domain, has found widespread applications in digital image processing. One of the pervasive transforms is the discrete cosine transform (DCT) firstly presented by Ahmed [1]. As is the discrete version of cosine function, DCT is an approximation of the Karhunen-Loeve transform (KLT) [2] and good at energy compaction. It is widely applied in image processing, such as image compression [3], image quality optimization [4] and image denoising [5].

Generally, eight types of DCTs all have excellent energy compaction properties. Applications of these eight DCTs are various. The types II and IV of DCTs are the most widely applied in many fields, such as image compression, image encryption [6], feature extraction [7], and image denoising [8]. However, float numbers of DCT matrices generated from the traditional ways always cost a lot of time. Therefore, many kinds of research are developing their factorizations and integer forms to diminish the computational cost, such as the lifting scheme-based binDCT [9], the binary DCT [10], the integer DCT with non-orthogonal structures [11], and the integer MDCT derived from DCT IV [12]. However, integer forms of the rest of DCTs are little investigated, even though they also have a lot of applications. For example, the DCTs I, V, and VIII can be implemented in the fractional Fourier transform [13].

On the other hand, 4×4 DCT matrices can be easier implemented and they can avoid some mismatch or ringing effects than other sizes during the transformation process [14][15]. Furthermore, they also can be expanded to other sizes of DCTs with factorization or block operations [16].

Therefore, in this paper, we design 4×4 parametric integer discrete cosine transforms (PIDCTs) to represent integer forms of DCTs II and IV, and other types of DCTs. In PIDCTs, the matrices are all composed by integer parameters and they are orthogonal parametric matrices which can represent different types of DCTs. To evaluate its coding performance, we make comparisons it with peer integer DCT and traditional DCTs in terms of transform coding gain and DCT distortion. The rest of this paper is organized as follows: Section II reviews all types of traditional DCTs, and proposes 4×4 integer DCT matrices with various parameters will be presented in Section III. Analysis results are shown in Section IV. Section V reaches conclusion and future work.

II. Traditional DCTs

We first review some N-order traditional Discrete Cosine Transforms (DCTs). The matrix of DCT III is the transpose of DCT II. The matrices of DCTs VI and VII can be deduced by DCT V. Therefore, here we lists the typical matrices of DCT types: DCTs I, II, IV, V, and VIII.

$$
\begin{aligned}
(C_N^I)_{i,j} &= \gamma_{i,j}^I \cos\left[\frac{ij\pi}{N-1}\right], \\
(C_N^{II})_{i,j} &= \gamma_{i,j}^{II} \cos\left[\frac{(2j+1)i\pi}{2N}\right], \\
(C_N^{IV})_{i,j} &= \frac{2\cos\left[\frac{(2i+1)(2j+1)\pi}{4N}\right]}{\sqrt{N}}, \\
(C_N^{V})_{i,j} &= \gamma_{i,j}^{V} \cos\left(\frac{2ij\pi}{2N-1}\right), \\
(C_N^{VIII})_{i,j} &= \frac{2\cos\left[\frac{(2i+1)(2j+1)\pi}{2(2N+1)}\right]}{\sqrt{2N+1}},
\end{aligned} \tag{1}
$$

Then the above $\gamma_{i,j}^I, \gamma_{i,j}^{II}$, and $\gamma_{i,j}^V$ are normalization constants, which can be defined as

$$
\begin{aligned}
\gamma_{i,j}^I &= \begin{cases} \sqrt{\frac{2}{N-1}}, & \text{if } i,j = 0, N-1 \\ \frac{2}{\sqrt{N-1}}, & \text{otherwise} \end{cases} \\
\gamma_{i,j}^{II} &= \begin{cases} \sqrt{\frac{1}{N}}, & \text{if } i = 0 \\ \sqrt{\frac{2}{N}}, & \text{otherwise} \end{cases} \\
\gamma_{i,j}^{V} &= \begin{cases} \sqrt{\frac{2}{2N-1}}, & \text{if } i,j = 0 \\ \frac{2}{\sqrt{2N-1}}, & \text{otherwise} \end{cases}
\end{aligned} \tag{2}
$$

where $0 \le i,j \le N-1$.

The inverse matrices of these types of DCTs are the transposes of their matrices since they are all orthogonal matrices.

III. Proposed 4×4 Parametric Integer Discrete Cosine Transforms

In this section, we present 4×4 parametric integer discrete cosine transforms (PIDCTs) to denote different types of DCTs matrices. It is shown that PIDCTs can represent the matrix of DCT II and those of other types of DCTs with integer parameters under some constraints. As is shown in Section II, PIDCTs denote the main types of DCTs in this section: DCTs I, II, IV, V, and VIII.

For a 4×4 type y of DCT specified as P^y in PIDCTs is defined as follows. When $y \in [1, 2, 4, 5, 8]$, P^y can be represented as

$$P^1(a_1, b_1) = \begin{bmatrix} a_1 & b_1 & b_1 & a_1 \\ b_1 & a_1 & -a_1 & -b_1 \\ b_1 & -a_1 & -a_1 & b_1 \\ a_1 & -b_1 & b_1 & a_1 \end{bmatrix}$$

$$P^2(a_2, b_2, c_2) = \begin{bmatrix} a_2 & a_2 & a_2 & a_2 \\ b_2 & c_2 & -c_2 & -b_2 \\ a_2 & -a_2 & -a_2 & a_2 \\ c_2 & -b_2 & b_2 & -c_2 \end{bmatrix},$$

where $2a_2^2 = b_2^2 + c_2^2$;

$$P^4(a_4, b_4, c_4, d_4) = \begin{bmatrix} a_4 & b_4 & c_4 & d_4 \\ b_4 & -d_4 & -a_4 & -c_4 \\ c_4 & -a_4 & d_4 & b_4 \\ d_4 & -c_4 & b_4 & -a_4 \end{bmatrix},$$

where $a_4b_4 - b_4d_4 - a_4c_4 - c_4d_4 = 0$;

$$P^5(a_5, b_5, c_5, d_5) = \begin{bmatrix} d_5^2 & d_5 & d_5 & d_5 \\ d_5 & a_5 & -b_5 & -c_5 \\ d_5 & -b_5 & -c_5 & a_5 \\ d_5 & -c_5 & a_5 & -b_5 \end{bmatrix},$$

where $d_5^2 + a_5 - b_5 - c_5 = 0$ and $d_5^2 - a_5b_5 - a_5c_5 + b_5c_5 = 0$;

$$P^8(a_8, b_8, c_8, d_8, e_8) = \begin{bmatrix} a_8 & b_8 & c_8 & d_8 \\ b_8 & -e_8 & -b_8 & -b_8 \\ c_8 & -b_8 & -d_8 & a_8 \\ d_8 & -b_8 & a_8 & -c_8 \end{bmatrix},$$

where $a_8 + e_8 - c_8 - d_8 = 0$ and $a_8c_8 - b_8^2 - c_8d_8 + a_8d_8 = 0$.

The PIDCT P^3 is the transpose of P^2 and it can denote the integer form of DCT III.

To be approximated to traditional DCTs, we need to obtain orthogonal forms of PIDCTs. P^y needs a normalization process with a quantization parameter $1/q_y$ as follows.

$$N^y = \frac{P^y}{q_y} \tag{3}$$

where q_y can be denoted as $q_1 = \sqrt{2(a_1^2 + b_1^2)}$, $q_2 = \sqrt{4a_2^2} = 2a_2$, $q_4 = \sqrt{a_4^2 + b_4^2 + c_4^2 + d_4^2}$, $q_5 = \sqrt{d_5^4 + 3d_5^2}$, and $q_8 = \sqrt{3b_8^2 + e_8^2}$, respectively. As is a unit orthogonal matrix N^y, the inverse of this matrix is its transposition.

All the parameters in PIDCTs are positive integers. Here we take some examples of PIDCTs. When $y = 1$, $a_1 = 2, b_1 = 3, q_1 = \sqrt{26}$, P^1 can be written as

$$P^1(2,3) = \begin{bmatrix} 2 & 3 & 3 & 2 \\ 3 & 2 & -2 & -3 \\ 3 & -2 & -2 & 3 \\ 2 & -3 & 3 & -2 \end{bmatrix}.$$

where $P^1(2,3)$ is an example of integer DCT I matrices.

When $y = 2, a_2 = 13, b_2 = 17, c_2 = 7, q = \sqrt{676}$, the integer matrix of DCT II can be shown by PIDCTs as follows

$$P^2(17,23,7) = \begin{bmatrix} 13 & 13 & 13 & 13 \\ 17 & 7 & -7 & -17 \\ 13 & -13 & -13 & 13 \\ 7 & -17 & 17 & -7 \end{bmatrix}.$$

One example of DCT IV can be represented with $q_4 = \sqrt{39}$ as

$$P^4(5,3,2,1) = \begin{bmatrix} 5 & 3 & 2 & 1 \\ 3 & -1 & -5 & -2 \\ 2 & -5 & 1 & 3 \\ 1 & -2 & 3 & -5 \end{bmatrix}.$$

Cases of DCT V and DCT VIII can be represented as following functions with $q_5 = \sqrt{2548}$ and $q_8 = \sqrt{36}$, respectively.

$$P^5(17,29,37,7) = \begin{bmatrix} 49 & 7 & 7 & 7 \\ 7 & 17 & -29 & -37 \\ 7 & -29 & -37 & 17 \\ 7 & -37 & 17 & -29 \end{bmatrix};$$

$$P^8(5,3,1,1,3) = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 3 & -3 & -3 & -3 \\ 1 & -3 & -1 & 5 \\ 1 & -3 & 5 & -1 \end{bmatrix}.$$

The quantization factor q_y can be moved to the quantization or initial process of image compression. Therefore, PIDCTs can avoid floating number operations since its matrices are composed of integers.

IV. Performance Analysis

In this section, we evaluate the performance of PIDCTs and compare it with other Integer cosine transform, e.g. LLM [17]. They will be compared in terms of DCT distortion and transform coding gain.

A. DCT Distortion

Most of integer DCTs are designed for imitating the traditional DCTs. The basis vectors of these integer DCTs are supposed to be approximate to that of traditional DCTs since they can obtain better transform coding performance. Mean square errors (MSEs) are usually used to estimate the distortion between integer and original DCTs. The smaller MSEs of integer transforms are closer to the original DCTs. Then we compare PIDCTs with LLM [17] to evaluate their MSEs with size of 4×4 matrices. Here some examples are listed as is shown in Table I.

TABLE I

MEAN SQUARE ERRORS OF PIDCTS AND LLM WITH THE SIZE OF 4×4.

Transform Types	Mean square errors
Traditional DCTs	0
LLM [17]	$7.979 * 10^{-2}$
$P^1(338, 239)$	**$1.703 * 10^{-11}$**
$P^2(13, 17, 7)$	$2.188 * 10^{-6}$
$P^4(145, 123, 82, 29)$	$1.116 * 10^{-6}$
$P^8(43, 38, 28, 16, 1)$	$2.4328 * 10^{-4}$

Table I presents the examples of PIDCTs are smaller than that of LLM, which means that PIDCTs are closer to traditional DCTs.

B. Coding Efficiency

An useful method to measure the performance of transform coding is to assess its transform coding gain G as following function.

$$G = 10\log_{10}\frac{1}{N}\sum_{i=0}^{N-1}\alpha_i^2/(\prod_{i=0}^{N-1}\alpha_i^2\|b_i\|^2)^{\frac{1}{N}}, \tag{4}$$

where N is the order of the transform, α_i^2 is the variance of the i^{th} transform coefficient, and $\|b_i\|^2$ is the 2-norm of i^{th} basis function of the transform matrix. The higher values of transform coding gain mean the more data the transform can compress. Table II displays the transform coding gains among traditional DCTs, the examples of LLM [17] and PIDCTs.

TABLE II

TRANSFORM CODING GAINS OF DIFFERENT INTEGER COSINE TRANSFORMS WITH THE SIZE OF 4×4.

Transform Types	Transform coding gain $G(dB)$				
	ρ = 0.7	ρ = 0.75	ρ = 0.8	ρ = 0.85	ρ = 0.9
DCT I	2.0633	2.5035	3.0475	3.7475	4.7195
$P^1(338, 239)$	2.0633	2.5035	3.0475	3.7475	4.7195
DCT II	2.1520	2.6524	3.2916	4.1453	5.3870
LLM [17]	2.1520	2.6517	3.2880	4.1382	5.3758
$P^2(265, 343, 151)$	**2.1533**	**2.6548**	**3.2935**	**4.1465**	**5.3873**
DCT IV	1.4122	1.6702	1.9658	2.3073	2.7064
$P^4(121, 120, 56, 44)$	1.4514	1.7562	2.1285	2.5991	3.2309
DCT V	2.0679	2.5284	3.1030	3.8477	4.8772
$P^5(1, 1, 1, 1)$	1.9486	2.4163	3.0187	3.8335	5.0342
DCT VIII	1.9015	2.2970	2.7757	3.3706	4.1423
$P^8(92, 80, 49, 44, 1)$	1.9291	2.3570	2.8907	3.5828	4.5429

As can be seen in Table II, the bold numbers represent biggest coding gain values of different DCTs, which are the results of our $P^2(265, 343, 151)$. Besides, it also shows a better transform coding gain than LLM since the matrix of LLM is DCT II. At the same time, other coding gains of PIDCTs are closer or even superior than corresponding traditional DCTs when α is belong to the set $\{0.7, 0.75, 0.8, 0.85, 0.9\}$.

C. Compression Performance

The compression process usually removes some redundancy information from images but it also disturbs some useful image data. To evaluate the performance of compression, the peak signal-to-noise ratio (PSNR) is an excellent tool to detect the difference between original and compressed images. Here we use a 512×512 color *pepper* image and compress it by traditional JPEG, LLM, and PIDCT.

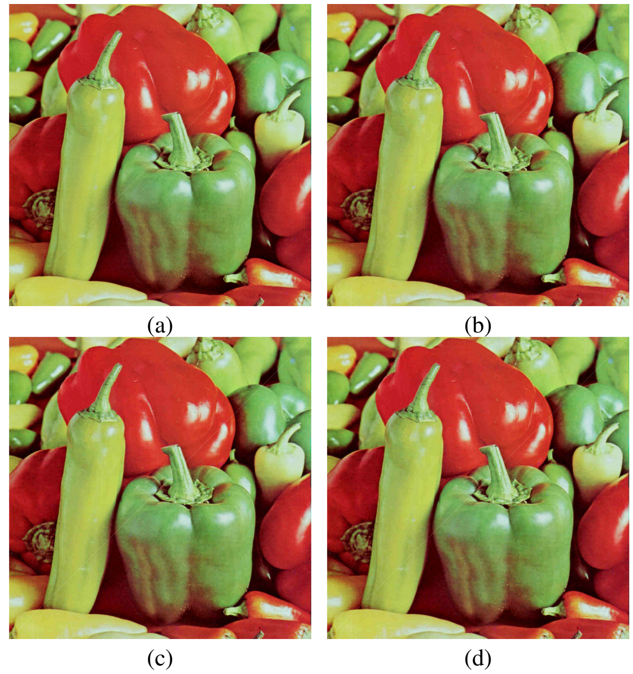

Fig. 1. (a) A color *pepper* image with 512×512 compressed by (b) JPEG (PSNR: 31.7629), (c) LLM (PSNR is 31.6998) and (d) PIDCT $P^2(39, 51, 21)$ (PSNR is 31.7630).

In Fig. 1 we can see that PIDCT has a better PSNR than traditional JPEG and LLM with the same quantization level. As one of the examples, it demonstrates that our PIDCT has a similar or better performance than traditional DCTs or other integer cosine transform.

V. CONCLUSION AND FUTURE WORK

As 4-order DCTs can avoid some ringing artifacts than other sizes, in this paper, we proposed a 4×4 parametric integer DCTs-like algorithm, i.e. PIDCTs. It cannot only imitate DCTs II and IV but also other types of DCTs since most research focused on integer forms of DCTs II and IV. Mainly PIDCTs can represent the matrices of 4-order integer DCTs I-IV, V, and VIII to avoid floating operations and reduce the computation cost. All the choices of coefficients of PIDCTs are flexible and parametric with some constraints. Transform evaluation results also demonstrate that PIDCTs are closer to traditional DCTs than peer integer DCT. It also shows a high transform efficiency in transform coding gain.

In future, we can expand PIDCTs to represent all types of DCTs to further develop its parametric property. Then we can explore more potentials of PIDCTs in different applications since different types of traditional DCTs have lots of valuable usages.

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